PRODUCT INTEGRAL SOLUTIONS FOR HEREDITARY SYSTEMS

BY

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ABSTRACT. Hereditary systems which satisfy a Lipschitz condition are solved in terms of product integrals. Realizations of this type of hereditary system are provided from functional differential and integral equations.

Introduction. Suppose that $\{X, \|\cdot\|\}$ is a Banach space, S is an interval of nonnegative numbers which includes 0, S' is an interval of numbers which contains S, and G is a linear space of functions from S' into X. We will use 0 to denote, where appropriate, either the real number, the zero element of X, or the member of G whose only value is the zero of X. Suppose that N is a function from S into the class of all pseudonorms on G such that

(i) If w is in S and f is in G then $N_w(f) = 0$ if and only if $f(u) = 0$, for all $u \leq w$,

(ii) if $u \leq w$ then $N_u(f) \leq N_w(f)$, for each f in G, and

(iii) $\{G, N\}$ is complete.

Let H denote the class of all functions from G into G and 1 the identity function in H. Suppose that K is a function from $S' \times S'$ into H such that

(i) if f is in G, $[u, v]$ is a subinterval of S', and t is a member of S' not exceeding u, then $[K(u, v)f](t) = 0$,

(ii) if each of $u, v,$ and w is in S' and v is either between u and w, $u = v$, or $v = w$, then $K(u, v)f + K(v, w)f = K(u, w)f$, for each f in G, and

(iii) there is a function k from S into the class of nondecreasing functions from S to the numbers such that, for each positive number w in S,

$$N_w(K(u, v)0) \leq k_w(v) - k_w(u)$$

and

$$N_w(K(u, v)f - K(u, v)g) \leq (L) \int_u^v N_x(f - g)\, dk_w(x),$$

for each (f, g) in $G \times G$ and subinterval $[u, v]$ of $[0, w]$.

We want to consider the equation

(1) $h(u) = f(u) + [K(0, u)h](u),$

where f is in G. Equation (1) will be called an *hereditary system*. The phrase
hereditary system is meant to suggest a system for which the response depends
only upon the previous history of the system. For Stieltjes integral equations
[7] the phrase well posed has been used to describe this situation. Amongst
functional differential and integral equations [3] those which exhibit this behavior
are called delay equations. Nonanticipatory is used for stochastic integral
equations [9]. Hale and Cruz [4] applied the phrase hereditary system to a class
of such systems described using derivatives. There is a large overlap of equation
(1) with the systems discussed in each of these areas. Our purpose is to apply
the method of product integration, which may be loosely described as a refine-
ment of the Cauchy-Euler method, to the problems of existence, uniqueness, and
continuity for hereditary systems.

   Existence and uniqueness. We should note that if h is in G then the func-
tion g from S' into X defined by $g(u) = [K(0, u)h](u)$, for all u in S', is in G.
This follows from the completeness of G. Let $\hat{g}$ denote the function from S into
G defined by $\hat{g}_u = K(0, u)h$, for each u in S. Then $N_w(\hat{g}_u - \hat{g}_v) = 0$, for u, v,
and w in S and $w \le u \le v$. Hence the net $\{S, \hat{g}\}$ has a limit g_1 in G. But
$g_1(u) = \hat{g}_w(u) = [K(0, w)h](u) = g(u)$, for w in S and u in S' not exceeding w.
So g is in G and $g(u) = 0$, for u in S' not exceeding 0.

   Thus for each f in G we seek a solution of equation (1) in G. Our method
for exhibiting such a solution is essentially that of [11] and depends upon the
introduction of a related operator equation. We will introduce and use Mac Nerney's
device [8] of jumping the dimension in order to circumvent the technical difficulty
that members of H might not map 0 to 0.

   Let $\mathcal{G} = G \times I(2)$, where $I(2)$ denotes the integers modulo 2. Define addition on $\mathcal{G}$
componentwise and $\mathcal{N}$ on $\mathcal{G}$ by $\mathcal{N}_u((g, m)) = N_u(g) + m$, for u in S and (g, m) in $\mathcal{G}$.
Let $\mathcal{H}$ denote the functions from $\mathcal{G}$ into $\mathcal{G}$ which map the zero of $\mathcal{G}$ to zero. We
will also use 1 to denote the identity function in $\mathcal{H}$. Let V denote the function
from $S \times S$ into H defined as follows:

$$[V(u, v)g](w) = \begin{cases} [K(v, u)g](w) & \text{if } v \le u, \\ 0 & \text{otherwise,} \end{cases}$$

for each g in G, (u, v) in $S \times S$, and w in S'. Let $\mathcal{U}$ denote the function from $S \times S$
into $\mathcal{H}$ defined by $\mathcal{U}(u, v)(g, m) = (V(u, v)g - V(u, v)0 + mV(u, v)0, 0)$, for each
(u, v) in $S \times S$ and (g, m) in $\mathcal{G}$.

   Suppose that k is a function from S into the class of nondecreasing functions
from S to the numbers corresponding to K as in condition (iii). Such a function k
will be called a *family of super functions* for K.

Theorem 1. *For each* (u, v) *in* $S \times S$ *and* g *in* G, *the continuously continued product* ${}_u\Pi^v\,[1 + V]g$ *is a member of* G.

Proof. The proof depends upon the recognition of the fact that our situation is only slightly more general than that encountered by Mac Nerney in [8]. There is no problem if $u \leq v$. Suppose that $[v, u]$ is a subinterval of S and each of (g_1, m_1) and (g_2, m_2) is in $\mathcal{G}$. If w is a positive number in S and $w \geq u$ then

$$\mathcal{N}_w(\mathcal{O}(u, v)(g_1, m_1) - \mathcal{O}(u, v)(g_2, m_2)) \leq \{k_w(u) - k_w(v)\}\mathcal{N}_w((g_1, m_1) - (g_2, m_2)).$$

From Lemma 1.2 of [8], we have that for each positive number b there is an order partition s of $\{u, v\}$ such that if t refines s then

$$\mathcal{N}_w\left(\prod_s [1 + \mathcal{O}](g, 1) - \prod_t [1 + \mathcal{O}](g, 1)\right) = N_w\left(\prod_s [1 + V]g - \prod_t [1 + V]g\right) < b.$$

Hence the existence of ${}_u\Pi^v\,[1 + V]g$ follows from the completeness of G.

Let W denote the function from $S \times S$ into H defined by $W(u, v)g = {}_u\Pi^v[1 + V]g$, for each (u, v) in $S \times S$ and g in G, and h the function from S into X defined by

$$h(u) = \begin{cases} f(u) & \text{if } u \leq 0, \\ [W(u, 0)f\,](u) & \text{if } 0 \leq u, \end{cases}$$

for each u in S'.

Theorem 2. h *is in* G *and* $[K(0, u)h](u) = [(R)\int_u^0 VW[\ , 0]f\,](u)$, *for each* u *in* S', *i.e.,* h *is a solution of equation* (1).

Proof. Let $\hat{g}$ denote the function from S into G defined by $\hat{g}(u) = W(u, 0)f$, for each u in S. The net $\{S, \hat{g}\}$ is Cauchy and $\hat{g}_u(v) = h(v)$, for each u in S and v in S' not exceeding u. Hence h is in G.

Let T be a positive number in S and $\mathcal{W}$ denote the function from $S \times S$ into $\mathcal{H}$ defined by $\mathcal{W}(u, v)(g, m) = {}_u\Pi^v\,[1 + \mathcal{O}](g, m)$, for each (u, v) in $S \times S$ and (g, m) in $\mathcal{G} \times \mathcal{G}$. We should note that $\mathcal{W}(u, v)(f, 1) = (W(u, v)f, 1)$, for each (u, v) in $S \times S$. Hence, for u, v, and w in $[0, T]$ with $u \leq v$, we have

$$N_w(W(v, 0)f - W(u, 0)f) = \mathcal{N}_w(\mathcal{W}(v, 0)(f, 1) - \mathcal{W}(u, 0)(f, 1))$$

$$= \mathcal{N}_w(\mathcal{W}(v, u)\mathcal{W}(u, 0)(f, 1) - \mathcal{W}(u, 0)(f, 1))$$

$$\leq (\mu(v, u) - 1)\mathcal{N}_w(\mathcal{W}(u, 0)(f, 1))$$

$$\leq (\mu(v, u) - 1)\mu(u, 0)(N_w(f) + 1)$$

$$= (\mu(v, 0) - \mu(u, 0))(N_w(f) + 1),$$

where $\mu(x, y) = {}_x\Pi^y [1 + dk_T]$ for each (x, y) in $S \times S$.

Suppose that u is in $[0, T]$ and ϵ is a positive number. There is an order partition r of $\{u, 0\}$ such that if $\{s_p\}_0^n$ refines r then

$$N_T\left((R) \int_u^0 VW[\ , 0]f - (R) \sum_s VW[\ , 0]f\right) < \frac{\epsilon}{3}$$

and

$$(R) \sum_t \mu[\ , 0]dk_T - (R) \sum_s \mu[\ , 0]dk_T < \frac{\epsilon}{3}(N_T(f) + 1),$$

for each refinement t of s. Let $\bar{s}(p) = s(n - p)$, for $p = 0, 1, 2, \cdots, n$, and t be a refinement of $\bar{s}$ such that if α is an increasing integer-valued sequence and $\bar{s} = t[\alpha]$ then

$$\sum_{p=1}^n (L) \int_{\bar{s}(p-1)}^{\bar{s}(p)} N_x(W(\bar{s}_{p-1}, 0)f - b) dk_T(x)$$

$$- \sum_{p=1}^n \sum_{q=\alpha(p-1)+1}^{\alpha(p)} N_{t(q-1)}(W(\bar{s}_{p-1}, 0)f - b)(k_T(t_q) - k_T(t_{q-1})) < \frac{\epsilon}{3}.$$

Thus

$$N_T\left((R) \int_u^0 VW[\ , 0]f - K(0, u)b\right)$$

$$\leq N_T\left((R) \int_u^0 VW[\ , 0]f - (R) \sum_s VW[\ , 0]f\right)$$

$$+ N_T\left(\sum_{p=1}^n K(s_p, s_{p-1})W(s_p, 0)f - K(0, u)b\right)$$

$$< \frac{\epsilon}{3} + N_T\left(\sum_{p=1}^n K(\bar{s}_{p-1}, \bar{s}_p)W(\bar{s}_{p-1}, 0)f - K(0, u)b\right)$$

$$\leq \frac{\epsilon}{3} + \sum_{p=1}^n (L) \int_{\bar{s}(p-1)}^{\bar{s}(p)} N_x(W(\bar{s}_{p-1}, 0)f - b) dk_T(x)$$

$$< \frac{2\epsilon}{3} + \sum_{p=1}^n \sum_{q=\alpha(p-1)+1}^{\alpha(p)} N_{t(q-1)}(W(\bar{s}_{p-1}, 0)f$$

$$- W(t_{q-1}, 0)f)(k_T(t_q) - k_T(t_{q-1}))$$

$$\leq \frac{2\epsilon}{3} + \left\{(L) \sum_t \mu[\ , 0]dk_T - (L) \sum_{\bar{s}} \mu[\ , 0]dk_T\right\}(N_T(f) + 1)$$

$$< \epsilon.$$

Therefore $N_v((R) \int_u^0 VW[\ , 0]f - K(0, u)h) = 0$ for each u and v in S and so $(R) \int_u^0 VW[\ , 0]f = K(0, u)h$, for each u in S. Hence $[(R) \int_u^0 VW[\ , 0]f](u) = [K(0, u)h](u)$, for each u in S.

Theorem 3. *Equation (1) has only one solution in* G.

Proof. Suppose that each of h_1 and h_2 is a solution of equation (1) in G and u is a positive number in S. Then

$$N_v(h_1 - h_2) = N_v(K(0, u)h_1 - K(0, u)h_2)$$

$$\leq (L) \int_0^v N_x(h_1 - h_2)\,dk_u(x) \leq N_u(h_1 - h_2)\{k_u(v) - k_u(0)\},$$

for each v in S not exceeding u. Therefore, for each positive integer n,

$$N_u(h_1 - h_2) \leq N_u(h_1 - h_2)[k_u(u) - k_u(0)]^n/n!$$

and so $N_u(h_1 - h_2) = 0$. This means $h_1 = h_2$.

Continuity. For each positive number t in S let μ_t denote the function from $S \times S$ to the nonnegative numbers defined by $\mu_t(u, v) = {}_u\Pi^v [1 + |dk_t|]$. Then we have for each positive number t in S, subinterval $[u, v]$ of S, and (g_1, g_2) in $G \times G$, that $N_t(W(u, v)g_1 - W(u, v)g_2) \leq \mu_t(u, v)N_t(g_1 - g_2)$. This observation serves as a proof for the following theorem.

Theorem 4. *If* $\{D, \hat{f}\}$ *is a Cauchy net in* G *with limit* f *and* $\hat{h}_n$, *for each* n *in* D, *is the member of* G *such that* $\hat{h}_n(u) = \hat{f}_n(u) + [K(0, u)\hat{h}_n](u)$, *for each* u *in* S, *then* $\{D, \hat{h}\}$ *has limit* h *in* G.

Theorem 5. *If* $S = [0, T]$; D *is a directed set;* $\hat{K}$ *is a function from* D *into the class of all functions from* $S \times S$ *into* H; *and* $\hat{k}$ *is a function from* D *into the set of nondecreasing functions on* S *such that* (1) $\hat{K}_n$ *satisfies* (i) *and* (ii) *above, for each* n *in* D, (2) $\hat{k}_n$ *is a super function for* $\hat{K}_n$ *for each* n *in* D, (g_1, g_2) *in* $G \times G$, *and subinterval* $[u, v]$ *of* S, (3) k_T *is the uniform limit of* $\{D, \hat{k}\}$; *and* (4) *for each positive number* b *and* g *in* G *there is a member* n *of* D *such that* $N_T(K(u, v)g - \hat{K}_m(u, v)g) < b$, *for each* m *in* D *which follows* n *and subinterval* $[u, v]$ *of* S; *then* $\{D, \hat{h}\}$, *where* $\hat{h}_n$ *for each* n *in* D *is the member of* G *such that* $\hat{h}_n(u) = f(u) + [\hat{K}_n(0, u)\hat{h}_n](u)$, *for each* u *in* S, *has limit* h *in* G.

Proof. Theorem 5 is a corollary of Theorem 3 [12]. For each n in D, let $\hat{F}_n$ denote the function from $S \times S$ into H defined by $[\hat{F}_n(u, v)g](w) = [\hat{K}_n(0, v)g](w)$. Let F denote the function from $S \times S$ into H defined by $[F(u, v)g](w) = [K(0, v)g](w)$. Then $\{D, \hat{F}\}$ is convergent with limit F. Hence $\{D, \hat{h}\}$ converges uniformly on S with limit h.

Realizations. The realizations of K presented in this section will demonstrate the wide variety of hereditary systems to which the methods of product integration apply. Suppose that $S = S'$, G is the class of quasicontinuous functions from S into X, and $N_u(f)$, for each u in S and f in G, is the least number b such that $\|f(x)\| \leq b$ for all x in S not exceeding u. Let M denote the class of all functions from X into X. Suppose that F is a function from $S \times S$ into M and k is a nondecreasing function from S to the nonnegative numbers such that (i) $F[\ , u]P$ is in G, for each u in S and P in X, (ii) $\|F(u, v)0 - F(u, w)0\| \leq |k(v) - k(w)|$ and

$$\|\{F(u, v) - F(u, w)\}P - \{F(u, v) - F(u, w)\}Q\| \leq |k(u) - k(v)| \cdot \|P - Q\|,$$

for all u, v, and w in S and (P, Q) in $X \times X$. The equation

$$(2) \qquad\qquad b(u) = f(u) + (L) \int_0^u dF[u, \ ]b,$$

where f is in G, was considered in [11]. We can relate those results to the present paper by introducing a function K from $S \times S$ into H as follows:

$$[K(u, v)g](t) = \begin{cases} (L) \displaystyle\int_u^v dF[t, \ ]g & \text{if } u \leq v \leq t, \\[3mm] (L) \displaystyle\int_u^t dF[t, \ ]g & \text{if } u \leq t \leq v, \\[3mm] 0 & \text{otherwise.} \end{cases}$$

With this realization, equations (1) and (2) are equivalent.

Mac Nerney [7] and Herod [5] have both introduced generalizations of Stieltjes integral equations in the direction of hereditary systems. Comparison of our paper with each of theirs might illuminate condition (iii) on K.

For our second realization, suppose that $X = R^n$, $\|\cdot\|$ is the Euclidean norm, $S' = S$, G is the class of quasicontinuous functions from S into X, and $N_u(f)$, for each u in S and f in G, is the least number b such that $\|f(x)\| \leq b$ for all x in S not exceeding u. Suppose that g is a function from $S \times S \times X$ into X and m is a function from $S \times S$ into the nonnegative numbers such that

(i) for each u in S and f in G, the function h from S into X defined by $h(v) = g(u, v, f(v))$ is Lebesgue summable on each compact subinterval of S,

(ii) for each subinterval $[u, v]$ of S and f in G the function $\int_u^v g[\ , s, f(s)]ds$ is in G,

(iii) $m[u, \ ]$ is summable on each compact subinterval of S, for each u in S,

(iv) $\|g(u, s, 0)\| \leq m(u, s)$ and $\|g(u, s, P) - g(u, s, Q)\| \leq m(u, s)\|P - Q\|$, for each u in S, almost all s in S, and (P, Q) in $X \times X$, and

(v) there is a nondecreasing function k from S to the nonnegative numbers such that $\int_u^v m(w, s)\,ds \leq k(v) - k(u)$, for each subinterval $[n, v]$ of S and w in S. The equation

(3) $$h(u) = f(u) + \int_0^u g(u, s, h(s))\,ds,$$

where f is in G, is seen to be equivalent to equation (1) upon defining K as follows:

$$[K(u, v)h](t) = \begin{cases} \int_u^v g(t, s, h(s))\,ds & \text{if } u \leq v \leq t, \\[2mm] \int_u^t g(t, s, h(s))\,ds & \text{if } u \leq t \leq v, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

Finally, suppose that $\{X, \|\cdot\|\}$ is either the space of real numbers or the space of complex numbers with the usual norm, r is a positive number, S is a compact interval $[0, T]$, and $S' = [-r, T]$. Let G be the class of all absolutely continuous functions from S' into X and let $N_u(f) = \text{l.u.b.}_{-r \leq x \leq u}\|f(x)\| + \int_{-r}^u \|df\|$. Suppose that m is a summable function with values in the nonnegative numbers, g_1 is a summable function with values in X, and g_2 is a function from $S \times [-r, 0]$ into X such that

(i) $g_2[u, l]$ is Lipschitz with Lipschitz constant $m(u)$, for each u in S, and

(ii) for each f in G, the function h from S into X defined by $h(u) = \int_{-r}^0 df[u + l]dg_2[u, l]/dl$ is summable.

The reader is referred to [2] for a discussion of the representation of continuous linear functionals on the space of absolutely continuous functions.

The equation

(4) $$h(u) = f(u) + \int_0^u g_1(s)h(s - r)\,ds + \int_0^u \left\{\int_{-r}^0 \frac{dh[s + l]dg_2[s, l]}{dl}\right\}ds,$$

where f is in G, is seen to be equivalent to equation (1) upon defining K as follows:

$$[K(u,v)h](t) = \begin{cases} \int_u^v g_1(s)h(s-r)\,ds + \int_u^v \left\{\int_{-r}^0 \frac{dh[s+l]dg_2[s,l]}{dl}\right\}ds & \text{if } u \leq v \leq t, \\[4mm] \int_u^t g_1(s)h(s-r)\,ds + \int_u^v \left\{\int_{-r}^0 \frac{dh[s+l]dg_2[s,l]}{dl}\right\}ds & \text{if } u \leq t \leq v, \\[4mm] 0 \quad \text{otherwise.} \end{cases}$$

Local theory. Condition (iii) on K is the critical assumption which permits the application of Mac Nerney's product integral theory to hereditary systems. Other researchers have studied the product integral [1], [10], [6] and this raises the possibility of weakening or changing condition (iii). A complete investigation of this problem is outside the scope of this paper; but in this section, two alternate versions of condition (iii) are presented which fit within the framework of the existence theory already developed.

We will assume in this section that, for each u in S and f in G, $N_u(f)$ is the least number b such that $\|f(x)\| \leq b$ for all x in S' not exceeding u. For each positive number b let T_b denote the function from X into X defined by

$$T_b P = \begin{cases} P & \text{if } \|P\| \leq b, \\ (b/\|P\|)P & \text{if } \|P\| \geq b, \end{cases}$$

for each P in X. We will assume, further, that $T_b[f]$ is a member of G, for each positive number b and f in G. Note that $N_u(T_b[f] - T_b[g]) \leq 2N_u(f - g)$, for each u in S, positive number b, and (f, g) in $G \times G$.

Suppose that K_1 is a function from $S' \times S'$ into H which satisfies conditions (i) and (ii) of the first section and

(iv) for each positive number b the function K from $S \times S$ into H defined by $K(u, v)g = K_1(u, v)T_b[g]$ has a family of continuous super functions.

Let V_1 denote the function from $S \times S$ into H defined as follows:

$$[V_1(u, v)f](w) = \begin{cases} [K_1(v, u)f](w) & \text{if } v \leq u, \\ 0 & \text{otherwise,} \end{cases}$$

for each f in G, (u, v) in $S \times S$, and w in S'.

Theorem 6. *For each f in G there is a positive number v in S such that, for each u in S not exceeding v, $W_1(u, 0)f = {}_u\Pi^0 [1 + V_1]f$ exists. Furthermore, if h is the function from S' into X defined by $h(u) = [W_1(v, 0)f](u)$, for all u in S', then h is in G and $h(u) = f(u) + [K_1(u, 0)h](u)$, for all u in S' not exceeding v.*

Proof. Suppose that w is a positive number in S, b is a number larger than $N_w(f) + 1$, k is a function from S into the class of continuous nondecreasing functions from S to the numbers corresponding to be as in condition (iv), and v is a positive number in $[0, w]$ such that $\exp(2k_w(v) - 2k_w(u)) \leq b/\{N_w(f) + 1\}$. Let V_b be the function from $S \times S$ into H defined by $V_b(u, v)f = V_1(u, v)T_b[f]$, for each (u, v) in $S \times S$ and g in G. Define $\mho_1$ and $\mho_b$ in terms of V_1 and V_b, respectively, as in §2. For each decreasing sequence $\{t_p\}_0^n$ such that $t_0 \leq v$ and $t_n = 0$, we have

$$\mathfrak{N}_w\left(\prod_t [1 + \mathcal{O}_b](f, 1)\right) \le \prod_t (1 + 2dk_w)\mathfrak{N}_w((f, 1))$$

$$\le \exp(2k_w(v) - 2k_w(u))(N_w(f) + 1) \le b.$$

Hence $N_w(\Pi_t [1 + V_b]f) \le b$. So, by induction, $\Pi_t [1 + V_b]f = \Pi_t [1 + V_1]f$ and hence $_u\Pi^0 [1 + V_1]f$ exists for each u in S not exceeding v. This also shows that $W_1(v, 0)f = W_b(v, 0)f$. If $b = W_1(v, 0)f$, then b is in G, $N_w(b) \le b$, and $b(u) = f(u) + [K_b(u, 0)b](u) = f(u) + [K_1(u, 0)b](u)$, for all u in S' not exceeding v.

For the remainder of the section, we will assume that G contains the constant functions and each member of G is continuous on S. Suppose that P is a point in X and K_2 is a function from $S' \times S'$ into H which satisfies conditions (i) and (ii) of the first section and

(v) there is a positive number b such that the function K from $S' \times S'$ into H defined by $K(u, v)g = K_2(u, v)\{T_b[g - 1_{s'}P] + 1_{s'}P\}$ has a family of continuous super functions.

Let V_2 denote the function from $S \times S$ into H defined as follows:

$$[V_2(u, v)f](w) = \begin{cases} [K_2(v, u)f](w) & \text{if } v \le u, \\ 0 & \text{otherwise,} \end{cases}$$

for each f in G, (u, v) in $S \times S$, and w in S'.

Theorem 7. *There is a positive number b such that if f is in G and $N_0(f - 1_{s'}P) < b$ then for some positive number v in S the product $W_2(u, 0)f = {}_u\Pi^0 [1 + V_2]f$ exists, for each u in S not exceeding v. Furthermore, if b is the function from S' into X defined by $b(u) = [W_2(v, 0)f](u)$, for all u in S', then b is in G and $b(u) = f(u) + [K_2(u, 0)b](u)$, for all u in S' not exceeding v.*

A proof may be constructed along the lines of the proof of the preceding theorem by introducing V_b as follows: $V_b(u, v)g = V_2(u, v)\{T_b[g - 1_{s'}P] + 1_{s'}P\}$, for (u, v) in $S \times S$ and g in G. The inequality [8]

$$N_v\left(\prod_t [1 + V_b]f - 1_{s'}P\right)$$

$$\le \mathfrak{N}_v\left(\sum_p \mathcal{O}_b(t_{p-1}, t_p) \prod_{q=p+1}^n [1 + \mathcal{O}_b(t_{q-1}, t_q)](f, 1)\right) + N_v(f - 1_{s'}P)$$

$$\le \{\exp(2k_w(v) - 2k(0)) - 1\}(N_v(f) + 1) + N_v(f - 1_{s'}P),$$

leads us to choose v so that

$$\{\exp(2k_w(v) - 2k_w(0)) - 1\}(N_v(f) + 1) + N_v(f - 1_{S'}P) \leq b.$$

Corollary. *If* $K_2(u, v)0 = 0$, *for each* (u, v) *in* $S' \times S'$, $P = 0$, *and* k *has only one value and that value has a bounded final set, then there is a positive number* d *such that if* f *is in* G *and* $N_w(f) \leq d$, *for each* w *in* S, *then* $W_2(u, 0)f = {}_u\Pi^0[1 + V_2]f$ *exists for each* n *in* S. *Furthermore, if* h *is the function from* S' *into* X *defined by*

$$h(u) = \begin{cases} f(u) & \text{if } u \leq 0, \\ [W_2(u, 0)f](u) & \text{if } 0 \leq u, \end{cases}$$

for each u *in* S', *then* h *is in* G *and* $h(u) = f(u) + [K_2(u, 0)h](u)$, *for all* u *in* S'.

More on realizations. Suppose that $X = R^n$, $\|\cdot\|$ is the Euclidean norm, r is a positive number, S is a compact interval $[0, T]$, $S' = [-r, T]$, G is the class of continuous functions from S' into X, and $N_u(f)$, for each u in S and f in G, is the least number b such that $\|f(x)\| \leq b$, for all x in S' not exceeding u. Let C denote the continuous functions from $[-r, 0]$ into X and $\|\cdot\|$, the supremum norm on C. For each u in S and f in G, let f_u denote the member of C defined by $f_u(x) = f(u + x)$, for all x in $[-r, 0]$.

Suppose that X_0 is an open subset of X, C_0 denotes the members of C with values in X_0, and g is a function from $X \times X \times C_0$ into X with the following properties.

  (i) For each f in G with values in X_0 and u in S the function h from S into X defined by $h(v) = g(u, v, f_v)$ is summable.

  (ii) For each subinterval $[u, v]$ of S and f in G with values in X_0 the function $\int_u^v g[\ , s, f_s]ds$ is in G.

  (iii) For each closed ball B contained in X_0 there is a function m from $S \times S$ to the nonnegative numbers and a continuous nondecreasing function k from S such that $\|g(u, s, f)\| \leq m(u, s)$ and $\|g(u, s, f) - g(u, s, g)\| \leq m(u, s)\|f - g\|_1$, for each u in S, almost all s in S, and f and g in C_0 with values in B; $m[u,]$ is summable, for each u in S; and $\int_u^v m(w, s)ds \leq k(v) - k(u)$, for each subinterval $[u, v]$ of S and w in S.

We want to relate the functional integral equation

(5) $$h(u) = \begin{cases} f(u) & \text{if } -r \leq u \leq 0, \\ f(u) + \int_0^u g(u, s, h_s)ds & \text{if } 0 \leq u, \end{cases}$$

where f is in C_0, to the development in the preceding section. Suppose that P is a point in X_0 and b is a positive number such that the closed ball B with

center P and radius r lies in X_0. Suppose that m and k are functions as in (iii), above. Let K_2 denote the function from $S' \times S'$ into H defined as follows:

$$[K_2(u, v)b](w) = \begin{cases} \int_x^y g(w, s, T_b[b_s - 1_{[-r,0]}P] + 1_{[-r,0]}P)ds & \text{if } u \leq v, \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $y = \max(0, \min(v, w))$ and $x = \max(0, \min(u, w))$.

If f in equation (5) satisfies $N_0(f - 1_{[-r,T]}P) \leq b$ then (5) is equivalent to $b(u) = f(u) + [K_2(0, u)b](u)$ and by Theorem 7 has a solution on $[-r, v]$ for some v in $[0, T]$ which can be represented as a product integral.

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